RESEARCH STATEMENT

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1. Introduction

My main research fields are analytic number theory, harmonic analysis and approximation theory. During the 4 years of my Ph.D. (2015-2019 at IMPA), I had my studies supervised by Emanuel Carneiro. During the Fall of 2018, I was a visiting scholar at the University of Mississippi supervised by Micah Milinovich. As part of my thesis, I worked in the following topics: (i) bandlimited approximations and applications in analytic number theory; (ii) omega results for functions in analytic number theory; (iii) linear programming bounds in analytic number theory.

The developments of these projects generated the following research articles:

(A1) *Bounding $S_n(t)$ on the Riemann hypothesis* (with E. Carneiro), Math. Proc. Camb. Philos. Soc. 164 (2018), 259–283.
(A2) *Bandlimited approximations and estimates for the Riemann zeta-function* (with E. Carneiro and M. B. Milinovich), Publ. Mat., to appear.
(A3) *A note on Entire L-functions*, Bull. Braz. Math. Soc., to appear.
(A4) *Extreme values for $S_n(\sigma, t)$ near the critical line*, Preprint Submitted arXiv:1807.11642.
(A5) *Pair correlation estimates for the zeros of the zeta-function via semidefinite programming* (with F. Gonçalves and D. de Laat), Preprint Submitted arXiv:1810.08843.
(A6) *Variance of primes in short intervals* (with E. Carneiro, V. Chandee and M. B. Milinovich), under final preparation.

In the following sections, I briefly describe these works, the motivation and current state of the area, my contributions, and the future projects that match my interests at this time.

2. Bandlimited approximations and applications in analytic number theory

In this section, I describe the research articles (A1), (A2), (A3) and (A6). As preliminaries in this section, we consider a particular instance of the so-called Beurling–Selberg extremal problem in Fourier analysis and approximation theory. In general terms, this is the problem of finding one-sided approximations of real-valued functions by entire functions of prescribed exponential type, seeking to minimize the $L^1(\mathbb{R})$-error. This problem has its origins in the work of A. Beurling in the late 1930's, in which he constructed extremal majorants and minorants of exponential type for the signum function. Later, A. Selberg used Beurling's extremal functions to produce majorants and majorants for characteristic functions of intervals and applied these in connection to large sieve inequalities. The survey [45] by J. D. Vaaler is the classical reference on the subject, describing some of the historical milestones of the problem and presenting several interesting applications of such special functions in analysis and number theory. In recent years there has been considerable progress both in the constructive aspects and in the range of applications of such extremal bandlimited approximations. For the constructive theory we highlight, for instance, the works [12, 13, 14, 31]. These allowed new applications in the theory of the Riemann zeta-function and general L-functions, for instance in [5, 6, 7, 8, 9, 10, 11, 16, 18, 23, 27, 37, 42].

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The *Gaussian subordination framework* of Carneiro, Littmann and Vaaler [13] is a robust method to solve the Beurling-Selberg problem for even functions in dimension one. In particular, functions $g : \mathbb{R} \to \mathbb{R}$ of the form

$$g(x) = \int_0^\infty e^{-\pi\lambda x^2} \mathrm{d}\nu(\lambda), \tag{2.1}$$

where ν is a finite nonnegative Borel measure on $(0, \infty)$, fall under the scope of [13].

2.1. **Bounds for the argument of the Riemann zeta-function and other L-functions.** In this subsection, I describe the articles (A1), (A2) and (A3). In the article (A1), we used the Gaussian subordination framework [13] to show bounds for objects related with the Riemann zeta-function. Let $\zeta(s)$ denote the Riemann zeta-function and let $S(t)$ denote the argument function that appears in the Von-Mangoldt formula for the distribution of the non-trivial zeros of $\zeta(s)$,

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right).$$

Useful information on the qualitative and quantitative behavior of $S(t)$ is encoded in its moments $S_n(t)$. Setting $S_0(t) = S(t)$ we define, for $n \geqslant 1$ and $t > 0$,

$$S_n(t) = \int_0^t S_{n-1}(\tau)\,\mathrm{d}\tau\ + \delta_n\,,$$

where δ_n is a specific constant depending on n. A classical result of Littlewood [35, Theorem 11] states that, under the Riemann hypothesis (RH),

$$S_n(t) = O\left(\frac{\log t}{(\log\log t)^{n+1}}\right) \tag{2.2}$$

for $n \geqslant 0$. The order of magnitude of (2.2) has not been improved over the last ninety years, and the efforts have since been concentrated in optimizing the values of the implicit constants. In the case $n = 0$, the best bound under RH is due to Carneiro, Chandee and Milinovich [7], who established that

$$|S(t)| \leqslant \left(\frac{1}{4} + o(1)\right)\frac{\log t}{\log\log t}. \tag{2.3}$$

The error term $o(1)$ is $O(\log\log\log t/\log\log t)$. This improved upon earlier works of Goldston and Gonek [27], Fujii [21] and Ramachandra and Sankaranarayanan [40]. For $n = 1$ the current best bound under RH is also due to Carneiro, Chandee and Milinovich [7], who showed that

$$-\left(\frac{\pi}{24} + o(1)\right)\frac{\log t}{(\log\log t)^2} \ \leqslant\ S_1(t)\ \leqslant\ \left(\frac{\pi}{48} + o(1)\right)\frac{\log t}{(\log\log t)^2}.$$

This improved upon earlier works of Fujii [22], and Karatsuba and Korolëv [34]. We established a new result for $n \geqslant 2$, under RH.

Theorem 1. *Assume the Riemann hypothesis. For $n \geqslant 0$ and t sufficiently large we have*

$$-\left(C_n^- + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}} \ \leqslant\ S_n(t)\ \leqslant\ \left(C_n^+ + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}}\,, \tag{2.4}$$

where $C_n^{\pm}$ are positive constants given by:

- *For $n = 4k+1$, with $k \in \mathbb{Z}^+$,*

$$C_n^- = \frac{\zeta(n+1)}{\pi\cdot 2^{n+1}} \quad \text{and} \quad C_n^+ = \frac{(1-2^{-n})\,\zeta(n+1)}{\pi\cdot 2^{n+1}}.$$

- *For $n = 4k+3$, with $k \in \mathbb{Z}^+$,*

$$C_n^- = \frac{(1-2^{-n})\,\zeta(n+1)}{\pi \cdot 2^{n+1}} \qquad \text{and} \qquad C_n^+ = \frac{\zeta(n+1)}{\pi \cdot 2^{n+1}}.$$

- *For $n \geqslant 2$ even,*

$$\begin{aligned} C_n^+ = C_n^- &= \left[\frac{2\big(C_{n+1}^+ + C_{n+1}^-\big)\ C_{n-1}^+\ C_{n-1}^-}{C_{n-1}^+ + C_{n-1}^-}\right]^{1/2} \\ &= \frac{\sqrt{2}}{\pi \cdot 2^{n+1}}\left[\frac{\big(1-2^{-n-2}\big)\ \big(1-2^{-n+1}\big)\ \zeta(n)\ \zeta(n+2)}{(1-2^{-n})}\right]^{1/2}. \end{aligned}$$

The terms $o(1)$ in (2.4) are $O(\log\log\log t/\log\log t)$.

This dramatically improves the previous best result of Wakasa [46]. In that paper he had established these inequalities with constants W_n (in place of our $C_n^{\pm}$) that tended to a stationary value of 0.3203... as $n \to \infty$. We highlight that our constants exponentially decaying. In fact, observe that $C_n^{\pm} \sim \frac{1}{\pi \cdot 2^{n+1}}$ when n is odd and large and $C_n^{\pm} \sim \frac{\sqrt{2}}{\pi \cdot 2^{n+1}}$ when n is even and large.

The outline of the proof of Theorem 1 is the following. The first step is to identify certain particular functions of a real variable naturally connected to the moments $S_n(t)$. Under RH, $S_n(t)$ can be expressed in terms of the sum of a translate of a certain function f_n over the ordinates of the non-trivial zeros of $\zeta(s)$. Since f_n is of class C^{n-1} but not higher (the n-th derivative of f_n is discontinuous at $x = 0$) it will be convenient to replace f_n by one-sided entire approximations of exponential type in a way that minimizes the $L^1(\mathbb{R})$-error. Here the machinery of bandlimited functions comes into play. For the case n odd, the function $f_n(x)$ can be expressed in the form (2.1), that allow us use appropriate majorants and minorants of exponential type for the function $f_n(x)$. We then apply a version of the Guinand–Weil explicit formula which connects the zeros of the zeta-function and the prime powers. When n is even, the solution of the Beurling–Selberg problem for these functions is quite a delicate issue and currently unknown. We are then forced to take a very different path in this case. Having obtained (2.4) for all odd n's, we proceed with an interpolation argument to obtain the estimate for the even n's in between, exploring the smoothness of $S_n(t)$ via the mean value theorem and solving two optimization problems.

Additionally, we obtained similar results to a general family of L-functions in the framework of [33, Chapter 5]. In particular, we extended the results of Carneiro, Chandee and Milinovich [8], and Carneiro and Finder [11] for $n \geqslant 2$.

In the article (A2), our main goal was to extend the bounds of Theorem 1 to the critical strip in an explicit way. The proof of this results follows the outline of the proof of Theorem 1. The novelty here is new majorants and minorants appear which represent major technical difficulties, because we need to work with the explicit form of their Fourier transforms. This implied a refinement in the estimates, compared with the proof of Theorem 1. In this way, we obtained a sharpened version of 1 with improved error terms (a factor $\log\log\log t$ has been removed). In particular, we improve the error term in the estimate (2.3) that is established in the following corollary.

Corollary 2. *Assume the Riemann hypothesis. For $t > 0$ sufficiently large we have*

$$|S(t)| \leqslant \frac{1}{4}\frac{\log t}{\log\log t} + O\left(\frac{\log t}{(\log\log t)^2}\right).$$

Another novelty was the study of the real part of the logarithmic derivative of the Riemann zeta-function. Using the lower bound that we obtained for this function, we also deduce a new proof of the optimal bound for $\log|\zeta(\frac{1}{2}+it)|$, under RH, that was established by Chandee and Soundararajan [16].

Corollary 3. *Assume the Riemann hypothesis. For $t > 0$ sufficiently large we have*

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| \leqslant \frac{\log 2}{2}\,\frac{\log t}{\log\log t} + O\left(\frac{\log t}{(\log\log t)^2}\right).$$

In the article (A3), my main goal was to extend the bounds obtained in (A2) for a family of entire L-functions. Additionally, I obtained bounds for the logarithm of these L-functions. In particular, the results in the critical line (see, for instance, [8, 9, 11, 16]) are recovered. In fact, I obtained a sharpened version of these results in the case of entire L-functions with improved error terms (a factor $\log\log\log C(t,\pi)^{\frac{3}{d}}$ has been removed).

2.2. **Distribution of primes in short intervals.** In this subsection, I describe the article (A6). Our main objective is to use new bounds for the second moment of the logarithmic derivative of $\zeta(s)$ to prove estimates for the variance of primes in short intervals. Our approach builds upon my previous work with Carneiro and Milinovich [10], incorporating ideas of Selberg [43] and Goldston, Gonek and Montgomery [28].

2.2.1. *The function $J(\beta, T)$.* For $\beta > 0$ and $T \geqslant 2$ we define

$$J(\beta,T) = \int_1^{T^\beta}\left(\psi\Big(x+\frac{x}{T}\Big) - \psi(x) - \frac{x}{T}\right)^2\frac{\mathrm{d}x}{x^2},$$

where $\psi(x) = \sum_{n\leqslant x}\Lambda(n)$, with $\Lambda(n)$ is the Von Mangoldt function. For $0 \leqslant \beta \leqslant 1$, Gallagher and Mueller [24] proved that

$$J(\beta,T) \sim \frac{\beta^2}{2}\frac{\log^2 T}{T}, \quad \text{as } T\to\infty.$$

Assuming the Riemann hypothesis, Selberg [43] showed that

$$J(\beta,T) = O_\beta\left(\frac{\log^2 T}{T}\right), \quad \text{as } T\to\infty, \tag{2.5}$$

for $1 \leqslant \beta \leqslant 4$, though his proof can modified to show that the estimate in (2.5) holds for each fixed $\beta \geqslant 1$. Montgomery (unpublished) proved a version of Selberg's result where the dependence on β is explicit. Assuming RH, for each $\beta \geqslant 1$, he proved that there is an absolute constant $C > 0$ such that

$$J(\beta,T) \leqslant C\cdot\frac{\beta\log^2 T}{T} \tag{2.6}$$

when T is sufficiently large. Alternate proofs of this result have been given in [24, 26, 28, 30]. We show the following result.

Theorem 4. *Assume the Riemann hypothesis. Then for each $\beta \geqslant 1$, we have*

$$J(\beta,T) \leqslant \big(4\beta - 1\big)\frac{\log^2 T}{T}$$

when T is sufficiently large.

For $\beta > 1$, it is unlikely that Theorem 4 can be replaced by an asymptotic formula using current methods. In fact, assuming RH, the combined work of Gallagher and Mueller [24] and Goldston [25] shows that asymptotic formula

$$J(\beta,T) \sim \left(\beta - \frac{1}{2}\right)\frac{\log^2 T}{T}, \quad \text{as } T\to\infty,$$

for each fixed $\beta \geqslant 1$ is equivalent to Montgomery's pair correlation conjecture for the zeros of the Riemann zeta-function.

The previous proofs of (2.6) in [24, 26, 28, 30] do not specify values of the admissible constant C, though the explicit value $C = 11$ can be deduced from the work of Goldston and Gonek in [26].

2.2.2. *The second moment of the logarithmic derivative of* $\zeta(s)$. To prove Theorem 4, we connect the function $J(\beta, T)$ with the second moment of the logarithmic derivative of $\zeta(s)$, near the critical line. Assuming RH, Selberg proved that for $\kappa > 0$ and $T \geqslant 4$ such that $e^{\kappa} = 1 + T^{-1}$, we have

$$\int_1^{\infty}\left(\psi\Big(x+\frac{x}{T}\Big)-\psi(x)-\frac{x}{T}\right)^2 e^{-2a\frac{\log x}{\log T}}\frac{\mathrm{d}x}{x^2} = \frac{4}{\pi}\int_0^{\infty}\left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2\left(\frac{\sin\frac{\kappa}{2}t}{t}\right)^2\mathrm{d}t + o(1).$$

uniformly for $a > 0$, as $T \to \infty$. Then, we need to study the behavior of the function

$$I(a,T) = \int_1^T\left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2\mathrm{d}t.$$

Goldston, Gonek and Montgomery [28] showed a relation between $I(a, T)$ and the integral of the function $F(\alpha)$ defined by Montgomery,

$$F(\alpha) := F(\alpha, T) = N(T)^{-1}\sum_{0<\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma'-\gamma)}w(\gamma'-\gamma),$$

where $\alpha \in \mathbb{R}$, $T \geqslant 2$, $N(T)$ is the number of zeros with ordinates $0 < \gamma \leqslant T$, $w(u) = 4/(4+u^2)$, and the sum is over all of pair of zeros of $\zeta(s)$. Assuming the Riemann hypothesis and that $T^{-1}\log^3 T \leqslant a \ll 1$, they showed that

$$I(a,T) = \left(\frac{1-e^{-2a}}{4a^2} + \int_1^{\infty}(F(\alpha)-1)e^{-2a\alpha}\mathrm{d}\alpha\right)(1+o(1))T\log^2 T$$

as $T \to \infty$. Using the bounds for the function $F(\alpha)$ in [26], Goldston, Gonek, and Montgomery showed an upper bound for the function $I(a, T)$ [28, Corollary 1]. We improve their result and establish the following theorem.

Theorem 5. *Assume the Riemann hypothesis and that* $T^{-1}\log^3 T \leqslant a \ll 1$. *Then we have, as* $T \to \infty$

$$I(a,T) \leqslant (1+o(1))\left(\frac{\coth a}{4a^2} - \frac{(\operatorname{csch} a)^2}{4a} + \frac{\coth a}{2} - \frac{1}{2}\right)T\log^2 T.$$

To prove this result, we bound the integral

$$\int_1^{\infty} F(\alpha)e^{-2a\alpha}\mathrm{d}\alpha$$

using the majorant of the Poisson kernel, that solves the Beurling-Selberg extremal problem associated. This majorant has already been used to obtain bounds for the real part of the logarithmic derivative of $\zeta(s)$ in [10].

2.3. **Future projects.**

2.3.1. *Weighted Hilbert's inequality.* Let $\{\delta_n\}_{n=1}^N$ a sequence of positive numbers and let $\{\lambda_n\}_{n=1}^N$ a sequence of real numbers such that $|\lambda_m - \lambda_n| \geqslant \delta_n$ for $m \neq n$. Let $\{a_n\}_{n=1}^N$ a sequence of complex numbers. The weighted Hilbert's inequality, proposed by Montgomery and Vaughan [38], establish that

$$\left|\sum_{\substack{m,n=1\\ m\neq n}}^{N}\frac{a_m\overline{a_n}}{\lambda_m-\lambda_n}\right| \leqslant \pi\sum_{n=1}^{N}\frac{|a_n|^2}{\delta_n}.$$

When $\delta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_m = m$ this inequality was proved by Hilbert (with constant 2π). This result was improved by Schur with sharp constant π. An alternative proof is due by Vaaler [45] using majorants and minorants for the signum function. The weighted Hilbert's inequality was proved by

Montgomery e Vaughan [38] with constant $\frac{3\pi}{2}$. The best constant up to date is due to Preissmann [39] with constant $\frac{4\pi}{3}$. Recently, Carneiro and Littmann (personal communication) obtained an alternative proof of the weighted Hilbert's inequality with constant 2π, using a majorant of the signum function with a special decay. Can we improve these ideas to obtain a better constant?

2.3.2. *Bandlimited radial decreasing approximations.* Similarly to the Beurling–Selberg problem, an interesting problem is to majorize some functions by bandlimited radial decreasing functions. For instance, we can think in majorize the delta function. Consider a real entire function $F : \mathbb{C}^n \to \mathbb{C}^n$ of exponential type 2π (with respect to the unit ball) such that its restriction to $\mathbb{R}^n$ is radial decreasing and $F(0) \geqslant 1$. What is the minimal value of $\widehat{F}(0)$? I studied the case $n = 1$ with Dimitar Dimitrov (UNESP), and we constructed some examples of these radial decreasing functions, based on a method of Sonin. These functions allow us obtaining good estimates for the answer in the case $n = 1$. Using the theory of Branges space, Carneiro (personal communication) solved the problem in the case $n = 2$. These functions can be interesting for some applications in number theory and have connections with the eigenvalues of the fractional Laplacian of the unit ball.

2.3.3. *Lower bounds for $J(\beta, T)$.* I am interested in showing a lower bound for $J(\beta, T)$. Using some results in [6] it is possible to obtain a lower bound for $I(a, T)$. Can we use some techniques of Goldston, Gonek, and Montgomery to derive the desired lower bound?

3. Omega results for functions in analytic number theory

In this section, I describe the article (A4). The main goal here was to obtain some extreme values for the functions $S_n(\sigma, t)$ in a region close to the critical line, extending the results in a recent paper of A. Bondarenko and K. Seip [2]. As a by-product, I obtained some omega results for the functions $S_n(t)$.

3.1. **Behavior in the critical line.** Omega results are inequalities that shows when certain quantity can attain very large values in a domain. We consider the following notations. The notation $f = \Omega_+(g)$ ($f = \Omega_-(g)$) means $f(t) > C\,g(t)$ ($f(t) < -C\,g(t)$) for some constant $C > 0$ and for some arbitrarily large values of t. The notation $f = \Omega_\pm(g)$ means that $f = \Omega_+(g)$ and $f = \Omega_-(g)$. Finally, the notation $f = \Omega(g)$ means that $\lim_{t\to\infty} f(t)/g(t) \neq 0$.

In 1977, Montgomery [36], under RH, established that

$$S(t) = \Omega_\pm\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right). \tag{3.1}$$

For the function $S_1(t)$, Tsang [44] established, under RH, that

$$S_1(t) = \Omega_\pm\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{3}{2}}}\right).$$

For the case $n \geqslant 2$, there are no known omega results for $S_n(t)$. Recently, A. Bondarenko and K. Seip used their version of the resonance method with a certain convolution formula for $\log\zeta(s)$ to produce large values of the functions $S(t)$ and $S_1(t)$.

Theorem 6 (cf. Bondarenko and Seip [2])**.** *Assume the Riemann hypothesis. Let $0 \leqslant \beta < 1$ be a fixed real number. Then there exist two positive constants c_0 and c_1 such that, whenever T is large enough,*

$$\max_{T^\beta \leqslant t \leqslant T} |S(t)| \geqslant c_0 \frac{(\log T)^{\frac{1}{2}}(\log\log\log T)^{\frac{1}{2}}}{(\log\log T)^{\frac{1}{2}}}$$

and

$$\max_{T^{\beta}\leqslant t\leqslant T} S_1(t) \geqslant c_1 \frac{(\log T)^{\frac12}(\log\log\log T)^{\frac12}}{(\log\log T)^{\frac32}}.$$

This result implies the following omega result for $S(t)$:

$$S(t) = \Omega\left(\frac{(\log t)^{\frac12}(\log\log\log t)^{\frac12}}{(\log\log t)^{\frac12}}\right).$$

This result can be compared with the $\Omega_{\pm}$ results of Montgomery (3.1). For $S_1(t)$, Theorem 6 improved the Ω_{+} result given by Tsang by a factor $(\log\log\log t)^{\frac12}$.

3.2. **Behavior in the critical strip.** For $\frac12 \leqslant \sigma \leqslant 1$ we define

$$S(\sigma,t) = \tfrac{1}{\pi}\arg\zeta\big(\sigma+it\big),$$

where the argument is obtained by a continuous variation along straight line segments joining the points 2, $2+it$ and $\sigma+it$, assuming that this path has no zeros of $\zeta(s)$, with the convention that $\arg\zeta(2)=0$. If this path has zeros of $\zeta(s)$ (including the endpoint $\sigma+it$) we set

$$S(\sigma,t) = \tfrac12 \lim_{\varepsilon\to 0^+}\left\{S(\sigma,t+\varepsilon)+S(\sigma,t-\varepsilon)\right\}.$$

Setting $S_0(\sigma,t)=S(\sigma,t)$ we define for $n\geqslant 1$ and $t>0$,

$$S_n(\sigma,t) = \int_0^t S_{n-1}(\sigma,\tau)\,\mathrm{d}\tau \; + \delta_{n,\sigma}, \tag{3.2}$$

where $\delta_{n,\sigma}$ is a specific constant depending on σ and n. Using the classical notation we have $S(\frac12,t)=S(t)$. In 1986, Tsang [44] stated, under RH the following lower bound

$$\sup_{t\in[T,2T]} \pm S(\sigma,t) \geqslant c\,\frac{(\log T)^{\frac12}}{(\log\log T)^{\frac12}}, \tag{3.3}$$

for $\frac12 \leqslant \sigma \leqslant \frac12 + \frac{1}{\log\log T}$, T sufficiently large and some constant $c>0$. This result shows extreme values for $S(\sigma,t)$ near the critical line. For the critical strip, a result of Montgomery [36] states that, for a fixed $\frac12 < \sigma < 1$, we have

$$S(\sigma,t) = \Omega_{\pm}\left((\sigma-\tfrac12)^2\frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}}\right). \tag{3.4}$$

In this article, I showed lower bounds for $S_n(\sigma,t)$ near the critical line, similar to (3.3).

Theorem 7. *Assume the Riemann hypothesis. Let $0\leqslant\beta<1$ be a fixed number. Let $\sigma>0$ be a real number and $T>0$ sufficiently large in the range*

$$\frac12 \leqslant \sigma \leqslant \frac12 + \frac{1}{\log\log T}.$$

Then there exists a sequence $\{c_n\}_{n\geqslant 0}$ of positive real numbers with the following property.

(1) *If $n\equiv 1 \mod 4$:*

$$\max_{T^{\beta}\leqslant t\leqslant T} S_n(\sigma,t) \geqslant c_n \frac{(\log T)^{1-\sigma}(\log\log\log T)^{\sigma}}{(\log\log T)^{\sigma+n}}.$$

(2) *In the other cases:*

$$\max_{T^{\beta}\leqslant t\leqslant T} |S_n(\sigma,t)| \geqslant c_n \frac{(\log T)^{1-\sigma}(\log\log\log T)^{\sigma}}{(\log\log T)^{\sigma+n}}.$$

Note that when $\sigma = \frac{1}{2}$ and $n = 0$ or 1, we recover Theorem 6. Moreover, we obtain new omega results on the critical line.

Corollary 8. *Assume the Riemann hypothesis. Then*

(1) *If* $n \equiv 1 \mod 4$*:*

$$S_n(t) = \Omega_+\left(\frac{(\log t \log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{n+\frac{1}{2}}}\right).$$

(2) *In the other cases:*

$$S_n(t) = \Omega\left(\frac{(\log t \log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{n+\frac{1}{2}}}\right).$$

The outline of the proof of Theorem 7 follows similar ideas of A. Bondarenko and K. Seip [2]. The main idea is to use the resonance method in an especial form due by Bondarenko and Seip. This implies the construction of a Dirichlet polynomial that allows finding large values for the function to study. The construction of our resonator is similar to that made by Bondarenko and Seip [2, Section 3]. Besides, we extend the convolution formula for $\log\zeta(s)$ in [44, Lemma 5] for the function $S_n(\sigma, t)$. A deeper analysis in some lemmas in [2] and [3] allows us to show these results for a region close to the critical line. To estimate the error terms I used some results in [10].

3.3. **Future projects.**

3.3.1. *New omega results.* I think it would be interesting to use these techniques of A. Bondarenko and K. Seip, to obtain other omega results. For example, to obtain the same results for $S_n(t)$ when $n \not\equiv 1 \mod 4$ with no absolute value. Besides, it would also be interesting to extend the region of Theorem 7 to the critical region, for improvement, in some sense, Montgomery's result in (3.4).

3.3.2. *Omega results for Dirichlet L-functions.* It will be interesting to extend these methods for obtaining better some results for Dirichlet L-functions. Using the results of B. Hough [32] and a recent pre-print of C. Aistleitner, K. Mahatab, M. Munsch and A. Peyrot [1] about omega results, it is possible to combine these techniques for improving the classical results?

4. Linear programming bounds in analytic number theory

In this section, I will describe the article (A5). The main goal here was to improve the asymptotic bounds for several quantities related to the zeros of the zeta-function (and other functions) under Montgomery's pair correlation approach [37]. We introduce a new technique to approach these problems, in which we replace the usual bandlimited auxiliary functions by the class of functions used in the linear programming bounds developed by Cohn and Elkies [19] for the sphere packing problem. The advantage of this framework is that it reduces the problems to convex optimization problems, which can be solved numerically via semidefinite programming. This technique usually produces better bounds than any bandlimited construction, and indeed this was the case for every problem we considered.

4.1. **Montgomery's pair correlation approach.** In 1973, Montgomery introduced a new framework for study some objects related with the distribution of zeros of the Riemann zeta-function.

4.1.1. *Pair correlation of zeros of the Riemann zeta-function.* For $T \geqslant 2$, we define the function

$$N^*(T) = \sum_{0<\gamma\leqslant T} m_\rho,$$

where the sum is over the non-trivial zeros of $\zeta(s)$ counting multiplicities. Using bandlimited functions, Montgomery showed that, assuming RH, $N^*(T) \leqslant (1.3333...+o(1))N(T)$. This result was later improved by Cheer and Goldston [17] to 1.3275. Assuming the generalized Riemann Hypothesis (GRH), Goldston, Gonek, Özlük and Snyder [29] improved it to 1.3262. We show the following result.

Theorem 9. *Assuming the Riemann hypothesis we have*

$$N^*(T) \leqslant (1.3208 + o(1))N(T), \quad \text{as } T \to \infty.$$

Assuming the Generalized Riemann hypothesis we have

$$N^*(T) \leqslant (1.3155 + o(1))N(T), \quad \text{as } T \to \infty.$$

An interesting application of these results is related to the percentage of distinct zeros. Let $N_d(T)$ be the number of distinct zeros of $\zeta(s)$ with ordinates $0 < \gamma \leqslant T$.

Theorem 10. *Assuming the Riemann hypothesis we have*

$$N_d(T) \geqslant (0.8477 + o(1))N(T), \quad \text{as } T \to \infty.$$

Assuming the Generalized Riemann hypothesis we have

$$N_d(T) \geqslant (0.8486 + o(1))N(T), \quad \text{as } T \to \infty.$$

Using the pair correlation approach, the best previous result known is due to Farmer, Gonek and Lee [20] with constant 0.8051. By a different technique, assuming RH, Bui and Heath-Brown [4] improved the constant to 0.8466.

Interested in the study of estimating how small the gaps between consecutive zeros can be related to the total average gap, Montgomery defined the pair correlation function

$$N(\beta, T) := \sum_{\substack{0<\gamma,\gamma'\leqslant T \\ 0<\gamma'-\gamma\leqslant \frac{2\pi\beta}{\log T}}} 1,$$

where $\beta > 0$, $T \geqslant 2$ and the sum runs over two sets of non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ and $\rho' = \frac{1}{2} + i\gamma'$ of $\zeta(s)$. An interesting problem related to this function is find the smallest $\beta > 0$ such that $N(\beta, T) \gg N(T)$. Montgomery established, under RH and $N(T) \sim N^*(T)$, that $N(0.68, T) \gg N(T)$. This result was improved by Goldston, Gonek, Özlük and Snyder [29] with constant 0.6072. Later, Carneiro, Chandee, Littmann and Milinovich [6] improved the constant to 0.6068.... Assuming GRH and $N(T) \sim N^*(T)$, Goldston, Gonek, Özlük and Snyder showed the constant 0.5781.... With our techniques we obtained

Theorem 11. *Assuming the Riemann hypothesis and $N(T) \sim N^*(T)$ we have*

$$N(0.6039, T) \gg N(T).$$

Assuming the Generalized Riemann hypothesis and $N(T) \sim N^*(T)$ we have

$$N(0.5769, T) \gg N(T).$$

4.1.2. *Pair correlation for Dirichlet L-functions.* Similar techniques allow improving a result for the distribution of zeros for the primitive Dirichlet L-functions. Using the approach by Chandee, Lee, Liu and Radziwill [15], we measure (in average) the proportion of simple zeros among all primitive Dirichlet L-functions assuming GRH. Let

$$N_{\Phi}(Q) := \sum_{Q \leqslant q \leqslant 2Q} \frac{W(q/Q)}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \text{primitive}}} \sum_{\gamma_\chi} \left|\mathcal{M}\Phi(i\gamma_\chi)\right|^2,$$

where W is a non-negative smooth function supported in $(1,2)$, the last sum is over all non-trivial zeros $\frac{1}{2} + i\gamma_\chi$ of the Dirichlet L-function $L(s,\chi)$, and $\mathcal{M}\Phi(s)$ is the Mellin transform of an special real-valued smooth function Φ supported in the interval $[a,b]$ with $0 < a < b < \infty$. Let

$$N_{\Phi,s}(Q) = \sum_{Q \leqslant q \leqslant 2Q} \frac{W(q/Q)}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \text{primitive}}} \sum_{\substack{\gamma_\chi \\ \text{simple}}} \left|\mathcal{M}\Phi(i\gamma_\chi)\right|^2.$$

Theorem 12. *Assuming the Generalized Riemann hypothesis we have*

$$N_{\Phi,s}(Q) \geqslant (0.9350 + o(1))N_{\Phi}(Q), \quad \text{as } Q \to \infty.$$

Using the pair correlation approach, the best previous result known is due to Sono [42], showing the number 0.9322....

4.1.3. *Pair correlation for $\xi'(s)$.* Farmer, Gonek and Lee [20] studied the pair correlation of zeros of the derivative of Riemann ξ-function. Using bandlimited functions they showed the proportion 0.8583 for the number of simple zeros of $\xi'(s)$. We improved their result, and we showed a new estimate for the distinct zeros of $\xi'(s)$. Defining the functions $N_1(T)$, $N_{1,s}(T)$ and $N_{1,d}(T)$ (quantity of zeros, simple zeros, and distinct zeros respectively) for $\xi'(s)$ as in the case of $\zeta(s)$ we obtain the following result.

Corollary 13. *Assuming RH we have*

$$N_{1,s}(T) \geqslant (0.8825 + o(1))N_1(T), \quad \text{as } T \to \infty.$$

and

$$N_{1,d}(T) \geqslant (0.9412 + o(1))N_1(T), \quad \text{as } T \to \infty.$$

4.2. **Future Projects.**

4.2.1. *Bounding objects related with the Riemann zeta-function via linear programming bounds.* In recent years, different applications to bandlimited functions of the Riemann zeta-function and general L-functions are showed, for instance, [5, 6, 7, 8, 9, 10, 11, 16]. This technique that uses semidefinite programming can help us replace (in some cases) the usual bandlimited auxiliary functions by other special functions. One particular problem that I would like to investigate is whether it is possible to improve the constant 1/4 in (2.3). An extension of the Gaussian subordination method [13] in a different space and similar interpolation formulas by Radchenko and Viazovska [41, Theorem 1] will be needed.

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